INTERNATIONAL BACCALAUREATE
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MARKSCHEME

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MATHEMATICS

Higher Level

Paper 2

30 pages

1. (a) Using integration by parts

$$\int x\cos 3x\,\mathrm{d}x = \frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\,\mathrm{d}x \qquad \textit{(M1)(A2)}$$

$$= \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C \quad (C \text{ not required}) \qquad \textit{(AG)}$$

[3 marks]

(b) (i) Area $= \left|\left[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right]_{\frac{\pi}{6}}^{\frac{3\pi}{6}}\right| = \frac{2\pi}{9}$ ***(M1)(A1)***

(ii) Area $= \left|\left[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right]_{\frac{3\pi}{6}}^{\frac{5\pi}{6}}\right| = \frac{4\pi}{9}$ ***(A1)***

(iii) Area $= \left|\left[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right]_{\frac{5\pi}{6}}^{\frac{7\pi}{6}}\right| = \frac{6\pi}{9}$ ***(A1)***

> **Note:** Accept negative answers for part (b), as long as they are exact.
> Do **not** accept answers found using a calculator.

[4 marks]

(c) The above areas form an arithmetic sequence with

$$u_1 = \frac{2\pi}{9} \text{ and } d = \frac{2\pi}{9} \qquad \textit{(A1)}$$

The required area $= S_n = \frac{n}{2}\left[\frac{4\pi}{9} + \frac{2\pi}{9}(n-1)\right]$ ***(M1)(A1)***

$$= \frac{n\pi}{9}(n+1) \qquad \textit{(A1)}$$

[4 marks]

Total [11 marks]

2. (a) Given the points A(−1, 2, 3), B(−1, 3, 5) and C(0, −1, 1),

then $\vec{AB} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \vec{AC} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$ ***(A1)***

and $|\vec{AB}| = \sqrt{5}, |\vec{AC}| = \sqrt{14}$ ***(A1)***

The size of the angle between the vectors $\vec{AB}$ and $\vec{AC}$ is given by

$\theta = \arccos\left(\frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}||\vec{AC}|}\right) = \arccos\left(\frac{-7}{\sqrt{5}\sqrt{14}}\right)$ ***(M1)***

$\theta = 147^\circ$ (3 s.f.) or 2.56 radians ***(A1)***

[4 marks]

(b) Area $= \frac{1}{2}|\vec{AB}||\vec{AC}| \sin\theta$ or $\frac{1}{2}|\vec{AB} \times \vec{AC}|$ ***(M1)***

Area $= 2.29$ units^2 $\left(\text{accept } 2.28, 2.30, \text{and } \frac{\sqrt{21}}{2}\right)$ ***(A1)***

[2 marks]

(c) (i) The parametric equations of l_1 and l_2 are

$l_1 : x = 2, \quad y = -1 + \lambda, \quad z = 2\lambda$ ***(A1)***

$l_2 : x = -1 + \mu, \quad y = 1 - 3\mu, \quad z = 1 - 2\mu$ ***(A1)***

Note: At **this** stage accept answers with the same parameter for both lines.

(ii) To test for a point of intersection we use the system of equations:

$$2 = -1 + \mu \quad ①$$
$$-1 + \lambda = 1 - 3\mu \quad ②$$
$$2\lambda = 1 - 2\mu \quad ③$$
(M1)

Then $\mu = 3, \ \lambda = -7$ from ① and ② ***(A1)***

Substituting into ③ gives RHS $= -14$, LHS $= -5$ ***(M1)***

Therefore the system of equations has no solution and the lines do not intersect.

[5 marks]

continued...

Question 2 continued

(d) The shortest distance is given by $\frac{|(\boldsymbol{e}-\boldsymbol{d})\cdot(\boldsymbol{l}_1\times\boldsymbol{l}_2)|}{|(\boldsymbol{l}_1\times\boldsymbol{l}_2)|}$ where $\boldsymbol{d}$ and $\boldsymbol{e}$ are the position vectors for the points D and E and where $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are the direction vectors for the lines $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$.

Then $\boldsymbol{l}_1\times\boldsymbol{l}_2=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 0 & 1 & 2\\ 1 & -3 & -2\end{vmatrix}=4\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$ ***(M1)(A1)***

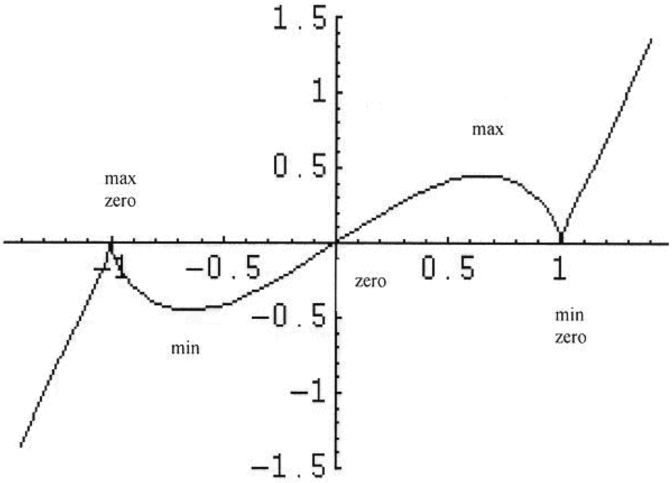
And $\frac{|(\boldsymbol{e}-\boldsymbol{d})\cdot(\boldsymbol{l}_1\times\boldsymbol{l}_2)|}{|(\boldsymbol{l}_1\times\boldsymbol{l}_2)|}=\frac{|(-3\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k})\cdot(4\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k})|}{\sqrt{21}}$ ***(M2)***

$=\frac{9}{\sqrt{21}}$ or 1.96 ***(A1)***

[5 marks]

Total [16 marks]

3. (a) $f(x) = x\left(\sqrt[3]{(x^2-1)^2}\right)$



(A4)

Notes: Award ***(A1)*** for the shape, including the two cusps (sharp points) at $x = \pm 1$.
(i) Award ***(A1)*** for the zeros at $x = \pm 1$ and $x = 0$.
(ii) Award ***(A1)*** for the maximum at $x = -1$ and the minimum at $x = 1$.
(iii) Award ***(A1)*** for the maximum at approx. $x = 0.65$, and the minimum at approx. $x = -0.65$
There are no asymptotes.
The candidates are not required to draw a scale.

[4 marks]

(b) (i) Let $f(x) = x(x^2-1)^{\frac{2}{3}}$

Then $f'(x) = \frac{4}{3}x^2(x^2-1)^{-\frac{1}{3}} + (x^2-1)^{\frac{2}{3}}$ ***(M1)(A2)***

$f'(x) = (x^2-1)^{-\frac{1}{3}}\left[\frac{4}{3}x^2 + (x^2-1)\right]$

$f'(x) = (x^2-1)^{-\frac{1}{3}}\left(\frac{7}{3}x^2 - 1\right)$ (or equivalent)

$f'(x) = \frac{7x^2-3}{3(x^2-1)^{\frac{1}{3}}}$ (or equivalent)

The domain is $-1.4 \le x \le 1.4, x \ne \pm 1$ (accept $-1.4 < x < 1.4, x \ne \pm 1$) ***(A1)***

(ii) For the maximum or minimum points let $f'(x) = 0$ *i.e.* $(7x^2 - 3) = 0$ or use the graph. ***(M1)***

Therefore, the x-coordinate of the maximum point is $x = \sqrt{\frac{3}{7}}$ (or 0.655) and ***(A1)***

the x-coordinate of the minimum point is $x = -\sqrt{\frac{3}{7}}$ (or –0.655). ***(A1)***

Notes: Candidates may do this using a GDC, in that case award ***(M1)(G2)***.

[7 marks]

continued...

Question 3 continued

(c) The x-coordinate of the point of inflexion is $x = \pm 1.1339$ ***(G2)***

OR

$f''(x) = \dfrac{4x(7x^2 - 9)}{9\sqrt[3]{(x^2-1)^4}}, x \neq \pm 1$ ***(M1)***

For the points of inflexion let $f''(x) = 0$ and use the graph, i.e. $x = \sqrt{\dfrac{9}{7}} = 1.1339$. ***(A1)***

Note:	Candidates may do this by plotting $f'(x)$ and finding the x-coordinate of the minimum point. There are other possible methods.

[2 marks]

Total [13 marks]

4. (i) Let p_n be the statement $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\cos x = \cos\left(x + \frac{n\pi}{2}\right)$ for all positive integer values of n.

For $n = 1, \frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$ ***(A1)***

$= \cos\left(x + \frac{\pi}{2}\right)$ ***(A1)***

Therefore p_1 is true.

Assume the formula is true for $n = k$,

that is, $\frac{\mathrm{d}^k}{\mathrm{d}x^k}(\cos x) = \cos\left(x + \frac{k\pi}{2}\right)$ ***(M1)***

Then $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}(\cos x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\cos\left(x + \frac{k\pi}{2}\right)\right)$

$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(\cos x) = -\sin\left(x + \frac{k\pi}{2}\right)$ ***(M1)***

$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(\cos x) = \cos\left(x + \frac{k\pi}{2} + \frac{\pi}{2}\right)$ ***(A1)***

$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(\cos x) = \cos\left(x + \frac{(k+1)\pi}{2}\right)$ ***(A1)***

which is p_n when $n = k + 1$.

(So if p_n is true for $n = k$ then it is true for $n = k + 1$ and by the principle of mathematical induction p_n is true for all positive integer values of n.) ***(R1)***

[7 marks]

continued…

Question 4 continued

(ii) (a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 10 & 3 \end{pmatrix}\begin{pmatrix} a \\ 2a \end{pmatrix} = \begin{pmatrix} 6a \\ 16a \end{pmatrix}$ (accept row vectors) ***(M1)***

Therefore the image point is $\mathrm{P}'(6a, 16a)$ ***(A1)***

[2 marks]

(b) From part (a), $y' = \frac{8}{3}x'$, therefore the equation of the image line is $y = \frac{8}{3}x$. ***(A2)***

[2 marks]

(c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 10 & 3 \end{pmatrix}\begin{pmatrix} a \\ ma \end{pmatrix} = \begin{pmatrix} 2a + 2ma \\ 10a + 3ma \end{pmatrix}$ ***(M1)***

Therefore the image point is $\mathrm{Q}'(2a + 2ma, 10a + 3ma)$ ***(A1)***

[2 marks]

(d) Since the image line has equation $y = mx$

$10a + 3ma = 2ma + 2m^2a$ ***(M1)***

$2m^2 - m - 10 = 0$

$(2m - 5)(m + 2) = 0$ ***(M1)***

$m = \frac{5}{2}, m = -2$ ***(A2)***

[4 marks]

Total [17 marks]

5. (a) $\mathrm{P}(X=3)=\left(\frac{3}{4}\right)^2\times\frac{1}{4}=\frac{9}{64}\ (=0.141 \text{ to } 3 \text{ s.f.})$ ***(M1)(A1)***

[2 marks]

(b) Let the probability of at least three misses before scoring twice $=\mathrm{P}(3\text{ m})$
Let S mean "Score" and M mean "Miss".

$\mathrm{P}(3\text{ m})=1-[\mathrm{P}(0\text{ misses})+\mathrm{P}(1\text{ miss})+\mathrm{P}(2\text{ misses})]$ ***(M1)***

$=1-[\mathrm{P}(\mathrm{SS})+\mathrm{P}(\mathrm{SMS}\text{ or }\mathrm{MSS})+\mathrm{P}(\mathrm{MMSS}\text{ or }\mathrm{MSMS}\text{ or }\mathrm{SMMS})]$ ***(M2)***

$=1-\left[\left(\frac{1}{4}\right)^2+2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)+3\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2\right]$ ***(A2)***

$=\frac{189}{256}\ (=0.738 \text{ to } 3 \text{ s.f.})$ ***(A1)***

[6 marks]

(c) $\mathrm{E}(x)=\sum_{\text{for all x}} x\mathrm{P}(x)=1\times\frac{1}{4}+2\times\frac{1}{4}\times\frac{3}{4}+3\times\frac{1}{4}\times\frac{3}{4}^2+\dots$ ***(M1)(A1)***

$=\frac{1}{4}\left(1+2\times\frac{3}{4}+3\times\frac{3}{4}^2+\dots\right)$ ***(A1)***

$=\frac{1}{4}\left(1-\frac{3}{4}\right)^{-2}$ (using the given result) ***(M1)***

$=\frac{1}{4}\left(\frac{1}{4}\right)^{-2}=\frac{1}{4}(4)^2=4$ ***(A1)(AG)***

[5 marks]

Total [13 marks]

6. (i) (a) Let X be the number of patients arriving at the emergency room in a 15 minute period. Rate of arrival in a 15 minute period $= \frac{15}{4} = 3.75$. *(M1)*

$$\mathrm{P}(X = 6) = \frac{(3.75)^6}{6!} \mathrm{e}^{-3.75}$$ *(M1)*

$$= 0.0908$$ *(A1)*

OR

$$\mathrm{P}(6 \text{ patients}) = 0.0908$$ *(G2)*

[3 marks]

(b) Let F_1, F_2 be random variables which represent the number of failures to answer telephone calls by the first and the second operator, respectively.

$F_1 \sim \mathrm{P}_0(0.01 \times 20) = \mathrm{P}_0(0.2)$. *(A1)*

$F_2 \sim \mathrm{P}_0(0.03 \times 40) = \mathrm{P}_0(1.2)$. *(A1)*

Since F_1 and F_2 are independent

$F_1 + F_2 \sim \mathrm{P}_0(0.2 + 1.2) = \mathrm{P}_0(1.4)$ *(M1)*

$\mathrm{P}(F_1 + F_2 \geq 2) = 1 - \mathrm{P}(F_1 + F_2 = 0) - \mathrm{P}(F_1 + F_2 = 1)$ *(M1)*

$= 1 - \mathrm{e}^{-1.4} - (1.4)\mathrm{e}^{-1.4} = 0.408$ *(A1)*

OR

$P(F_1 + F_2 \geq 2) = 0.408$ *(M0)(G2)*

[5 marks]

(ii) (a) **Test statistics**: Difference of two sample means *t*-test is used, as sample sizes are small. *(M1)*

Variance: We use pooled variance, s^2_{n+m-2} where

$$s^2_{n+m-2} = \frac{ns_n^2 + ms_m^2}{n + m - 2}.$$ *(A1)*

Reason: The two sampled populations are normally distributed with equal population variances (and the sample is "small"). *(R1)*

[3 marks]

continued...

Question 6 (ii) continued

(b) $s^2_{n+m-2} = \dfrac{13\times1.8^2 + 15\times1.6^2}{13+15-2} = 3.097$ ***(A1)***

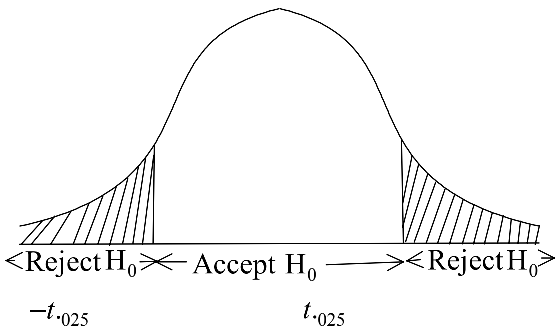
$H_0 : \mu_1 - \mu_2 = 0$
$H_1 : \mu_1 - \mu_2 \neq 0$ ***(A1)***

$$t = \frac{(6.8-5.3)-0}{\sqrt{(3.097)\left(\frac{1}{13}+\frac{1}{15}\right)}}$$ ***(M1)***

$= 2.25$ ***(A1)***

v = the number of degrees of freedom $= 13+15-2 = 26$ ***(A1)***

At 5% level of significance, the acceptance and rejection regions are shown:
$t_{0.025}$ with 26 degrees of freedom is 2.056



(M1)

Since the computed value of $t = 2.25$ falls in the rejection region, we reject H_0 and conclude that there is a difference between the population means. ***(R1)***

[7 marks]

(c) 99% confidence interval is $(6.8-5.3) \pm 2.779\sqrt{3.097\left(\frac{1}{13}+\frac{1}{15}\right)}$

$= (-0.353, 3.35)$ ***(M1)***

Since zero lies in the 99% confidence interval we accept the null hypothesis that there is no significant difference. ***(R1)***

[2 marks]

continued...

Question 6 continued

(iii) (a) **Test**: χ^2 test for independence, **test statistic**: Chisquare statistic ***(A1)***

[1 mark]

(b) Combining the last two columns, we have the following table of information:

Number of cups	sleep is worse	sleep is the same or better	Total
1	5	10	15
2	10	5	15
3	25	5	30
Total	40	20	60

(M1)(A1)

H_0 : There is no difference in sleeping pattern.
H_1 : There is a difference in sleeping pattern. ***(A1)***

Table of expected frequencies are:

Number of cups	sleep is worse	sleep is the same or better	Total
1	**10**	**5**	15
2	**10**	**5**	15
3	**20**	**10**	30
Total	40	20	60

(A2)

Note: Award ***(A2)*** for 5 or 6 correct bold entries.
Award ***(A1)*** for 3 or 4 correct, ***(A0)*** for 2 or less.

Number of degrees of freedom $= (2-1)(3-1) = 2$. ***(A1)***

$\chi^2_{0.05}$ with 2 degrees of freedom = 5.99. ***(A1)***

Computed value of Chi-square is given by

$$\chi^2 = \frac{(5-10)^2}{10} + \frac{(10-5)^2}{5} + \frac{(10-10)^2}{10} + \frac{(5-5)^2}{5} + \frac{(25-20)^2}{20} + \frac{(5-10)^2}{10}$$
$$= 11.25$$

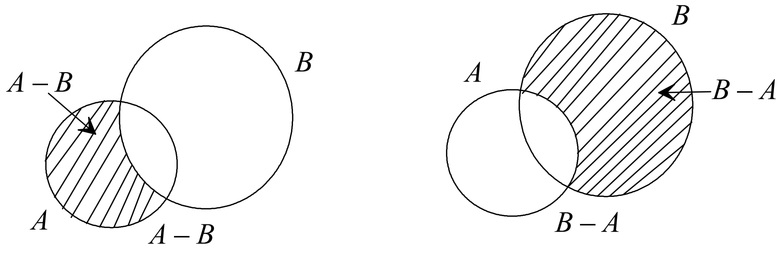
(M1)

Since, 11.25, the calculated value of $\chi^2 > 5.99$, the critical value, we reject the null hypothesis. Hence there is evidence that drinking coffee has an effect on sleeping pattern. ***(R1)***

[9 marks]

Total [30 marks]

7. (i) Venn diagrams are



(A1)

Note: Award *(A1)* if both the Venn diagrams are correct otherwise award *(A0)*.

From the Venn diagrams, we see that
$B \cap (A - B) = \phi$ and $B \cap (B - A) = B - A$ *(M1)*
Hence they are not equal. *(C1)*

Note: Award *(M0)(C1)* if no reason is given. Accept other correct diagrams.

[3 marks]

(ii) A relation R is defined on $\mathbb{Z} \times \mathbb{Z}^+$ by:

$$(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}^+, (a, b) R (c, d) \Leftrightarrow ad = bc .$$

(a) To show that R is an equivalence relation, we show that it is reflexive, symmetric and transitive.

Reflexivity: Since $ab = ba$ for $a, b \in \mathbb{Z}$, we have $(a, b) R (a, b)$. *(A1)*

Symmetry: $(a, b) R (c, d) \Leftrightarrow ad = bc \Leftrightarrow da = cb \Leftrightarrow cb = da$
$\Leftrightarrow (c, d) R (a, b)$ *(A1)*

Transitivity: $(a, b) R (c, d)$ and $(c, d) R (e, f) \Rightarrow ad = bc$ and $cf = ed$.
If $c = 0$, $ad = 0$ and $ed = 0$. Since $d \neq 0$, $a = 0$ and $e = 0$. *(M1)*
$\Rightarrow af = be \Rightarrow (a, b) R (e, f)$.
If $c \neq 0$, $adcf = bced$ i.e. $(af)dc = (be)cd$ or $(af)cd = (be)cd$
i.e. $af = be \Rightarrow (a, b) R (e, f)$, since $cd \neq 0$ *(R1)*

Note: Award *(M0)(R1)* if $cd \neq 0$ is not mentioned.

[4 marks]

(b) $ad = bc \Leftrightarrow a : b = c : d$ *(M1)*

i.e. the classes are those pairs (a, b) and (c, d) with $\frac{a}{b} = \frac{c}{d}$

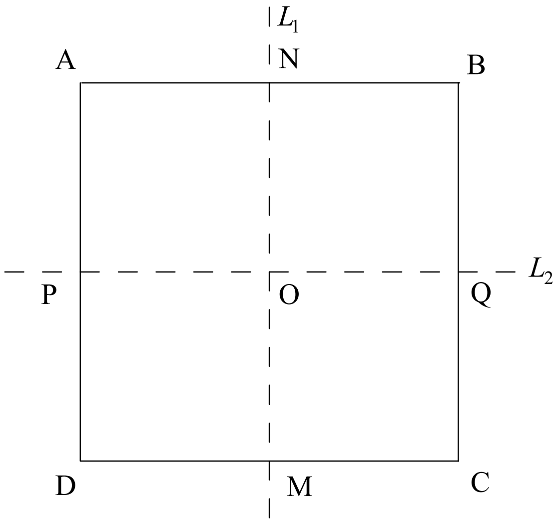
i.e. the elements of those pairs are in the same ratio.
i.e. the elements are on the same line going through the origin. *(R1)*

[2 marks]

continued...

Question 7 continued

(iii)



(a)

$\circ$	U	H	V	K
U	U	H	V	K
H	H	U	K	V
V	V	K	U	H
K	K	V	H	U

(A4)

Note: Award ***(A4)*** for 15 or 16 correct entries, ***(A3)*** for 13 or 14, ***(A2)*** for 11 or 12, ***(A1)*** for 9 or 10, ***(A0)*** for 8 or fewer.

[4 marks]

(b) Closure: U, H, K and V are the only entries in the table. So it is closed. ***(A1)***

Identity: U, since $UT = TU = T$ for all T in S. ***(A1)***

Inverses: $U^{-1} = U$, $H^{-1} = H$, $V^{-1} = V$, $K^{-1} = K$ ***(A1)***

Associativity: Given ***(AG)***

Hence $(S, \circ)$ forms a group. ***(R1)***

[4 marks]

continued...

Question 7 continued

(c) $C = \{1, -1, \text{i}, -\text{i}\}$

◊	1	–1	i	–i
1	1	–1	i	–i
–1	–1	1	–i	i
i	i	–i	–1	1
–i	–i	i	1	–1

(A3)

Note: Award ***(A3)*** for 15 or 16 correct entries, ***(A2)*** for 13 or 14, ***(A1)*** for 11 or 12, ***(A0)*** for 10 or fewer.

[3 marks]

(d) Suppose $f : S \to C$ is an isomorphism.
Then $f(U) = 1$, the identity in *C*, since *f* preserves the group operation. ***(M1)(C1)***
Assume $f(H) = \text{i}$, $1 = f(U) = f(H \circ H) = f(H) \lozenge f(H)$. ***(A1)***
But $f(H) = \text{i}$, and i is not its own inverse, so *f* is not an isomorphism. ***(R1)***

Note: Accept other correctly justified solutions.

[4 marks]

(iv) Given $(G, *)$ is a cyclic group with identity e and $G \neq \{e\}$ and G has no proper subgroups.
If *G* is of composite finite order and is cyclic, then there is $x \in G$ such that x generates *G*. ***(R1)***
If $|G| = p \times q$, $p, q \neq 1$, then $< x^p >$ is a subgroup of *G* of order q which is impossible since *G* has no non-trivial proper subgroup. ***(M1)*** ***(R1)***
Suppose the order of *G* is infinite. Then $< x^2 >$ is a proper subgroup of *G* which contradicts the fact that *G* has no proper subgroup. ***(M1)*** ***(A1)***
So *G* is a finite cyclic group of prime order. ***(R1)***

[6 marks]

Total [30 marks]

8. (i) Given $a_{n+2} = a_{n+1} + 2a_n \quad (n \geq 2), a_0 = 1, a_1 = 5$

The characteristic equation is $r^2 - r - 2 = 0 \Rightarrow (r-2)(r+1) = 0$ ***(M1)***

Therefore $r = 2$ or $r = -1$ ***(A1)***

The general solution is given by

$a_n = A2^n + B(-1)^n$ ***(A1)***

Using $a_0 = 1$ and $a_1 = 5$, we have,

$\begin{cases} A + B = 1 \\ 2A - B = 5 \end{cases}$ ***(M1)***

Hence, $A = 2$ and $B = -1$ ***(A1)***

Required solution: $a_n = 2^{n+1} + (-1)^{n+1}$ ***(R1)***

[6 marks]

(ii) By prime factorization of the integers *a* and *b*, there are primes $p_1, p_2, \ldots, p_n$ and non-negative integers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ such that

$a = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_n^{a_n}$, and $b = p_1^{b_1} \times p_2^{b_2} \times \ldots \times p_n^{b_n}$. ***(M1)***

Hence $\gcd(a,b) = p_1^{\min(a_1,b_1)} \times p_2^{\min(a_2,b_2)} \times \ldots \times p_n^{\min(a_n,b_n)}$ ***(A1)***

and $\operatorname{lcm}(a,b) = p_1^{\max(a_1,b_1)} \times p_2^{\max(a_2,b_2)} \times \ldots \times p_n^{\max(a_n,b_n)}$ ***(M1)***

Therefore $\gcd(a,b) \times \operatorname{lcm}(a,b) = p_1^{(a_1+b_1)} \times p_2^{(a_2+b_2)} \times \ldots \times p_n^{(a_n+b_n)}$ ***(M1)***

$= \left(p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_n^{a_n}\right)\left(p_1^{b_1} \times p_2^{b_2} \times \ldots \times p_n^{b_n}\right)$ ***(A1)***

$= a \times b$ ***(AG)***

[5 marks]

(iii) (a) Let *f* be the number of faces. By Euler's formula $v - e + f = 2$. ***(M1)***

Every edge bounds at most two faces and every face is bounded by at least three edges.

Hence, $e \geq \frac{3}{2} f$ or $3f \leq 2e$. ***(R1)(A1)***

From Euler's formula and $f \leq \frac{2}{3} e$,

$2 = v - e + f \leq v - e + \frac{2}{3} e = v - \frac{1}{3} e$ ***(M1)***

$\Rightarrow 2 \leq v - \frac{e}{3} \Rightarrow 6 \leq 3v - e \Rightarrow e \leq 3v - 6$. ***(A1)***

[5 marks]

(b) κ_5 has 5 vertices and 10 edges. ***(C1)***

From $e \leq 3v - 6$, we get, $10 \leq 15 - 6 = 9$, which is impossible and hence ***(M1)***

κ_5 is not a planar graph. ***(R1)***

[3 marks]

continued...

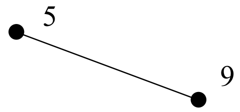
Question 8 continued

(iv) 5 is the root of the tree.

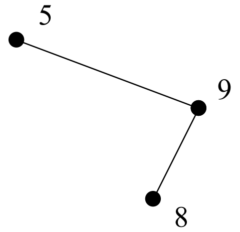
List | **Method** | **Construction**

5

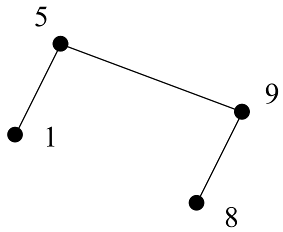
9 — 9 > 5, Root so we go right ***(M1)***



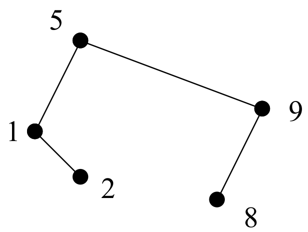
8 — 8 < 9 but still > 5, go left ***(M1)***



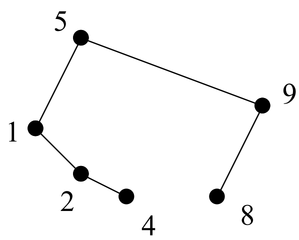
1 — 1 < 5, so go left from 5 ***(M1)***



2 — 2 >1 but less than 5, go right from 1 ***(M1)***



4 — 4 > 2 but still < 5 and > 1, so go right from 2 ***(A1)***



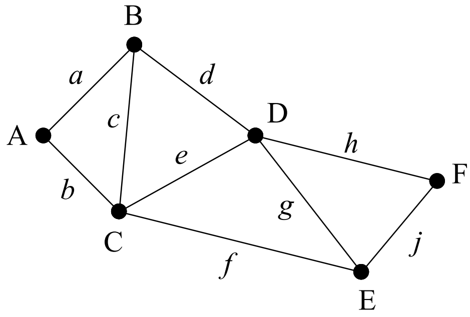
Note:	If candidates explain their working and then draw the final binary tree, award marks accordingly.

[5 marks]

continued...

Question 8 continued

(v)



Breadth – first search algorithm from A
Visit all 'depth 1' first, then 'depth 2' *etc.* ***(M1)***
L is a set of vertices, T is a set of edges.

Label	Set L	Set T	
0	{A}	ϕ	***(M1)***
1	{A, B, C}	$\{a, b\}$	***(M1)***
2	{A, B, C, D, E}	$\{a, b, e, f\}$ or $\{a, b, d, f\}$	***(M1)***
3	{A, B, C, D, E, F}	$\{a, b, e, f, h\}$ or $\{a, b, e, f, j\}$ or $\{a, b, d, f, h\}$ or $\{a, b, d, f, j\}$	***(M1)***

Spanning trees

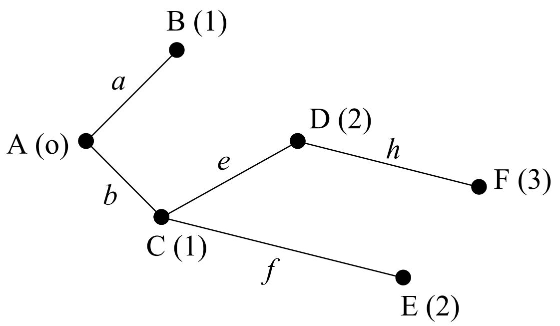


Diagram 1

OR

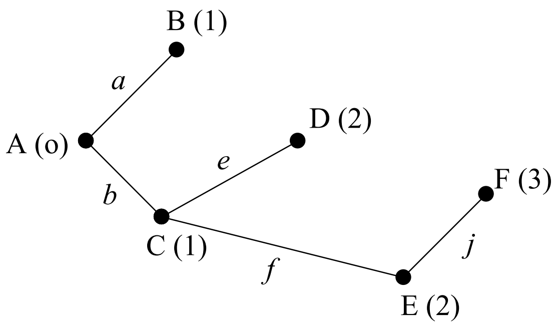


Diagram 2

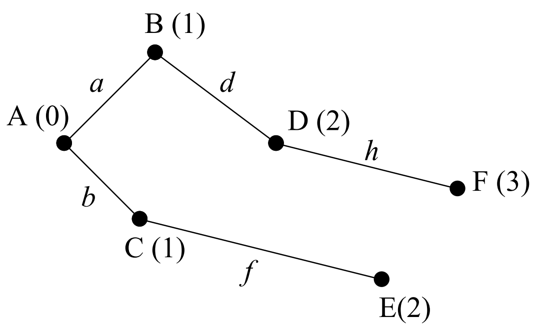


Diagram 3

OR

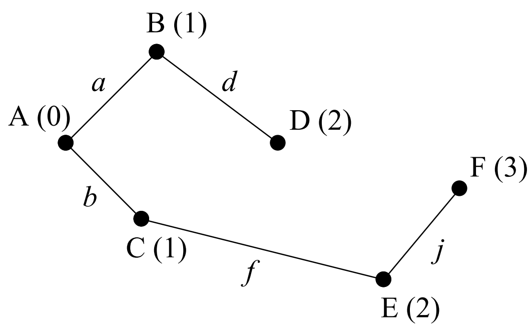


Diagram 4

(A1)

Note: Award ***(A1)*** for any one correct diagram. Accept other correct solutions with reasoning.

[6 marks]
Total [30 marks]

9. (i) (a) Mean Value Theorem: If f is continuous on $[a, b]$ and differentiable on $]a, b[$, *(A1)*
then there exists c in (a, b) such that $f(b) - f(a) = (b-a)f'(c)$. *(A1)*

> **Note:** Award *(A0)* for any error on the assumptions of f.
> Award *(A0)* if there is any error in the conclusion.
> Do not penalise if c in $]a, b[$ or $a < c < b$ is not mentioned.
> Accept integral form of the mean value thereon.

[2 marks]

(b) Consider $f(u) = u^k, u \geq 0, 0 < k \leq 1$. *(M1)*
Take $a = 1, b = 1 + x$.
By mean value theorem there exists 'c' between a and b so that
$f'(c) = \dfrac{(1+x)^k - 1^k}{1+x-1} \Rightarrow (1+x)^k - 1 = xkc^{k-1}$. *(M1)*
Therefore $(1+x)^k = 1 + kxc^{k-1}$. *(A1)*
For $x \geq 0$, c between 1 and $1+x$ implies $c \geq 1$. *(M1)*
For $0 < k \leq 1, -1 < k-1 \leq 0$.
Hence, $c^{-1} < c^{k-1} \leq c^0$ implies $0 < c^{k-1} \leq 1$, *(A1)*
Therefore $kxc^{k-1} \leq kx$ *(R1)*
$\Rightarrow 1 + kxc^{k-1} \leq 1 + kx$ *(A1)*
$\Rightarrow (1+x)^k \leq 1 + kx$ *(AG)*

[7 marks]

(ii) Error term for Simpson's rule is $-\dfrac{(b-a)h^4}{180} f^4(c)$ for some c in $]a, b[$, $h = \dfrac{b-a}{2n}$.

For $\displaystyle\int_2^7 \frac{\mathrm{d}x}{x}$, we have $b - a = 5$, *(M1)*

$f(x) = x^{-1}, f^4(x) = (-1)^4 \dfrac{4!}{x^5}$ *(A1)*

Maximum | error | $= \left(\dfrac{5}{2n}\right)^4 \left(\dfrac{5}{180}\right) \dfrac{24}{2^5} < 5 \times 10^{-5}$ $\left(\text{accept } 10^{-4}\right)$ *(M1)*

Therefore $n^4 > \left(\dfrac{5}{2}\right)^4 \times \dfrac{1}{180} \times \dfrac{24}{2^5} \times 10^5 = \dfrac{1250}{768} \times 10^4 \Rightarrow n > 1.13 \times 10 = 11.3$. *(A1)*

Hence take $n = 12$.

Therefore h, the step size, is $\dfrac{5}{24} = 0.208$ (3 s.f.) *(A1)*

[5 marks]

continued...

Question 9 continued

(iii) (a) $f(x) = \ln(1+x)$, $f(0) = 0$

$f'(x) = \dfrac{1}{1+x}$, $f'(0) = 1$ ***(A1)***

$f''(x) = -(1+x)^{-2}$, $f''(0) = -1$ ***(A1)***

$f'''(x) = (-1)(-2)(1+x)^{-3}$, $f'''(0) = 2$ ***(A1)***

$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$, $f^{(n)}(0) = (-1)^{n-1}(n-1)!$

Maclaurin's series for $f(x) = \ln(1+x)$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$$ ***(A1)***

Note:	Award ***(A0)*** if the general term $(-1)^{n-1}\dfrac{x^n}{n}$ is not written.

[4 marks]

(b) First $(n+1)$ terms give $R_n = \dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c such that $0 < c < x$.

On substitution $R_n = \dfrac{(-1)^n n!\, x^{n+1}}{(n+1)!\,(1+c)^{n+1}} = \dfrac{(-1)^n x^{n+1}}{(n+1)(1+c)^{n+1}}$, ***(M1)***

$$|R_n| = \frac{x^{n+1}}{(n+1)(1+c)^{n+1}} < \frac{1}{(n+1)} \text{ for } 0 \le x < 1,$$ ***(AG)***

since $0 < c < x$. ***(A1)***

Notes:	Award ***(A0)*** if the reasons $0 < c < x$, $0 \le x < 1$ are not written. Accept an answer using estimation of error in an alternating series.

[2 marks]

continued...

Question 9 continued

(iv) **Note:** Do not accept unjustified answers, even if correct.

(a) Compare the series with $\sum_{n=1}^{\infty} \frac{1}{n}$. ***(M1)***

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1$$ ***(M1)(A1)***

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ is divergent by the comparison test. ***(M1)(A1)***

[5 marks]

(b) $\cos n\pi = (-1)^n$

Hence $u_n = \frac{(n+10)\cos n\pi}{n^{1.4}} = (-1)^n \frac{(n+10)}{n^{1.4}} = (-1)^n v_n$ ***(C1)***

with $v_n = \frac{n+10}{n^{1.4}}$

$\Rightarrow \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^n v_n$, ***(M1)***

$v_n = \frac{n+10}{n^{1.4}}$ is a decreasing sequence in n ***(M1)***

$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{n+10}{n^{1.4}} = \lim_{n\to\infty} \frac{1}{n^{0.4}} = 0$, ***(A1)***

so the series $\sum_{n=1}^{\infty} \frac{(n+10)\cos n\pi}{n^{1.4}}$ is convergent, by the alternating series test. ***(R1)***

[5 marks]

Total [30 marks]

Note: There might be inconsistencies in the Markscheme depending on the diagram drawn. Do **not** penalize candidates for incorrect or non-use of brackets.

10. (i) (a) Given a triangle ABC. Let [AD], [BE], [CF] be such that D lies on [BC], E lies on [CA] and F lies on [AB]. ***(A1)***

Ceva's theorem: If [AD], [BE] and [CF] are concurrent, then

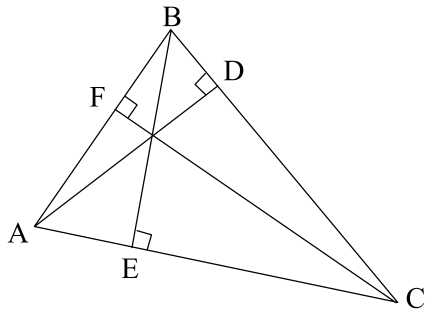
$$\frac{AF}{FB}\times\frac{BD}{DC}\times\frac{CE}{EA}=1$$ ***(A1)***

Converse (corollary) of Ceva's theorem:

$\frac{AF}{FB}\times\frac{BD}{DC}\times\frac{CE}{EA}=1$ then [AD], [BE] and [CF] are concurrent. ***(A1)***

[3 marks]

(b)



Let (AD), (BE) and (CF) be the altitudes of ΔABC.

ΔADB and ΔCFB are similar, since $A\hat{B}C$ is common and the triangles are right angled triangles. So $\frac{BF}{DB}=\frac{CF}{AD}$ (1) ***(M1)***

Similarly, from right triangles AEB and AFC,

$$\frac{AE}{FA}=\frac{EB}{CF}$$ (2) ***(M1)***

Also from right triangles CEB and CDA,

$$\frac{CD}{EC}=\frac{AD}{EB}$$ (3) ***(M1)***

From (1), (2), (3),

$$\frac{BF}{DB}\times\frac{AE}{FA}\times\frac{CD}{EC}=\frac{CF}{AD}\times\frac{EB}{CF}\times\frac{AD}{EB}$$ ***(M1)***

$$\Rightarrow\frac{AE}{EC}\times\frac{CD}{DB}\times\frac{BF}{FA}=1$$ (4) ***(A1)***

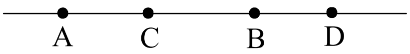
By the converse of Ceva's theorem (AD), (BE) and (CF) are concurrent. ***(R1)***

[6 marks]

continued...

Question 10 continued

(ii) (a)



A, B, C, D divide the line [AB] in harmonic ratio if $\frac{\text{AC}}{\text{BC}} = \frac{\text{AD}}{\text{DB}}$. ***(A2)***

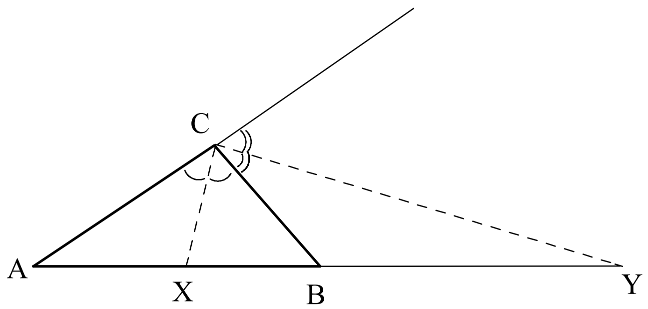
OR

C and D divide [AB] internally and externally in the same ratio *i.e.*

$$\frac{\text{AC}}{\text{CB}} = -\frac{\text{AD}}{\text{DB}}$$ ***(A2)***

[2 marks]

(b)



Given ΔABC. Let (CX) and (CY) be the internal and external angle bisectors of the angle ACB. ***(M1)***

By the angle bisector theorem $\frac{\text{AX}}{\text{XB}} = \frac{\text{AC}}{\text{CB}}$ and $\frac{\text{AY}}{\text{BY}} = \frac{\text{AC}}{\text{CB}}$ ***(M1)(M1)***

Therefore $\frac{\text{AX}}{\text{XB}} = -\frac{\text{AY}}{\text{YB}}$ ***(R1)***

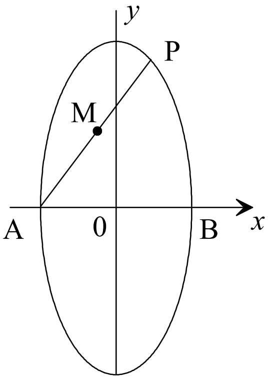
and A, X, B, Y are in harmonic ratio. ***(R1)***

[5 marks]

continued...

Question 10 continued

(iii)



Ellipse is $9x^2 + 4y^2 = 36$.

So A is (–2, 0). ***(A1)***

Let P be (x, y) and M be (α, β).

Then, $\alpha = \frac{1}{2}(-2 + x)$, $\beta = \frac{1}{2}(0 + y)$. ***(A1)(A1)***

Since P is on the ellipse $9x^2 + 4y^2 = 36$

$9(2\alpha + 2)^2 + 4(2\beta)^2 = 36 \Rightarrow 9(\alpha + 1)^2 + 4\beta^2 = 9$ ***(M1)***

Locus of M is $9(x+1)^2 + 4y^2 = 9$ ***(A1)***

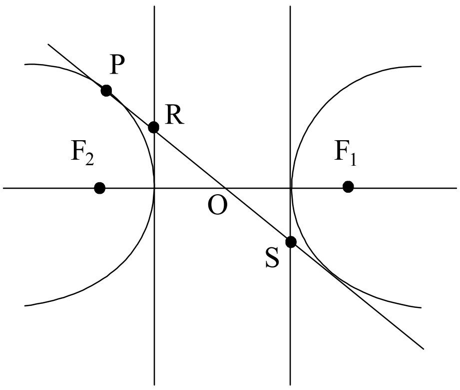
It is an ellipse with centre (–1, 0) and semiaxes 1 and $\frac{3}{2}$ (or equivalent). ***(R1)***

[6 marks]

continued...

Question 10 continued

(iv)



Note:	Please note that O is not the origin. (RS) is not necessarily tangential to the right-hand branch of the hyperbola.

Hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Let P be (x_1, y_1), $y\ , \neq 0$

Tangent to hyperbola at (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ ***(A2)***

Tangent at the vertices are $x=\pm a$. ***(A1)***

Hence R and S have coordinates $\left(\pm a, \left(\pm\frac{x_1}{a}-1\right)\frac{b^2}{y_1}\right)$, respectively. ***(M1)***

Therefore the midpoint of RS is $\left(0, -\frac{b^2}{y_1}\right)$. ***(A1)***

Take O as the midpoint of RS. Let the foci F_1 and F_2 be $(\pm c, 0)$ with $c^2=a^2+b^2$.
We shall show that $OR=OS=OF_1=OF_2$ and conclude that R, S, F_1, F_2 lie on a circle with centre O and radius OR.

Note that $OF_1{}^2=OF_2{}^2=c^2+\frac{b^4}{y_1{}^2}$ (1) ***(M1)***

Also $OR^2=(a-0)^2+\left[\left(\frac{x_1}{a}-1\right)\frac{b^2}{y_1}+\frac{b^2}{y_1}\right]^2$

$=a^2+\frac{x_1{}^2b^4}{y_1{}^2a^2}$ (2) ***(M1)***

Using $\frac{x_1{}^2}{a^2}-\frac{y_1{}^2}{b^2}=1$ in (2), by substituting for $\frac{x_1{}^2}{a^2}$, we get

$OS^2=OR^2=a^2+\left(1+\frac{y_1{}^2}{b^2}\right)\frac{b^4}{y_1{}^2}=a^2+b^2+\frac{b^4}{y_1{}^2}=c^2+\frac{b^4}{y_1{}^2}$. ***(A1)***

and the points R, S, F_1, F_2 lie on a circle. ***(AG)***

Note:	Award ***(R2)*** to candidates who worked out the case when P is on (F_1F_2) and the circle is a straight line.

[8 marks]
Total [30 marks]